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On the existence of Cohen extensions and $\sum_3^1$ predicates I

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In the present paper, we shall consider only Cohen extensions that do not use notions of forcing which are proper classes in a given model. From now on, accoding to Takahashi [13], this kind of Cohen extensions we will call Cohenian extensions.

Let $\mathcal{L}$ be the first-order language with the equality symbol "=" and the membership relation symbol "$\in$", but without other non-logical symbols. We use ZF for the Zelmelo-Fraenkel axiom system (extensionality, regularity, infinity, union, replacement and power set) that is formulated in $\mathcal{L}$, and ZFC for ZF plus the axiom of choice formulated in $\mathcal{L}$.

Suppose that $\mathfrak{M}$ is a countable standard transitive model for ZF. For each set m of $\mathfrak{M}$, we choose a constant symbol $\underline{m}$ called the name of m. It is understood that different names are chosen for different sets. The language obtained from $\mathcal{L}$ by adding all names of sets in $\mathfrak{M}$ is denoted by $\mathcal{L}_{\mathfrak{M}}$.

We shall consider the following problem: Let φ be a sentence of $\mathcal{L}_{\mathfrak{M}}$. Then can we find a Cohenian extension of $\mathfrak{M}$ that satisfies φ ?

*) The auther is in Dr. M. Takahashi's debt for several useful suggestions. Also he gave me that Solovay obtained a simple proof of Takahashi's theorem, but I could not know his proof.

Since there is a sentence of $\mathcal{L}$ that is not true in arbitrary structure for $\mathcal{L}$, some restriction on φ is necessary in order to answer our problem. Now let us consider only φ for which there is a countable standard transitive model $\mathfrak{N}$ for <u>ZF</u> that is an extension of $\mathfrak{M}$ having the same ordinals as $\mathfrak{M}$, and that satisfies φ .

Now let us also suppose that $\mathfrak{M}$ is one of Easton's model([1]) that satisfies the statement: <u>for every regular cardinal</u> $\aleph_\sigma$, $2^{\aleph_\sigma} < \aleph_\sigma$. Jensen[3] constructs a countable standard trasitive model $\mathfrak{N}$ that is an extension of $\mathfrak{M}$ having the same ordinals as $\mathfrak{M}$, and that satisfies <u>GCH</u>(the generalized continuum hypothesis). Jensen's construction of his model $\mathfrak{N}$ uses a notion of forcing that is a proper class of $\mathfrak{M}$. We can not construct his model $\mathfrak{N}$ using a notion of forcing which is a set of $\mathfrak{M}$, for, in arbitrary Cohenian extension, the collapsed cardinals constitute only a set of the Cohenian extension(Cf. Jech[2]). Thus answer to our problem is still negative.

Lévy [5] shows that <u>GCH</u> is a $\prod\frac{ZF}{2}$ sentence in his hierarchy of set theoretic formulas. This suggests that φ must be restricted to either $\sum\frac{ZF}{1}$ or $\prod\frac{ZF}{1}$ sentence of $\mathcal{L}_{\mathfrak{M}}$.

Let φ be $\prod\frac{ZF}{1}$ sentence of $\mathcal{L}_{\mathfrak{M}}$ such that there is a standard transitive extension $\mathfrak{N}$ of $\mathfrak{M}$ that satisfies φ . Then we have that $\mathfrak{M}$ also satisfies φ , for the $\prod\frac{ZF}{1}$ sentences of $\mathcal{L}_{\mathfrak{M}}$ are preserved between $\mathfrak{M}$ and $\mathfrak{N}$. Thus φ is satisfied in the trivial Cohenian extension $\mathfrak{M}$ of $\mathfrak{M}$ (use, as a notion of forcing, a linearly ordered structure in $\mathfrak{M}$). This give us an affirmative answer to our problem when φ is a $\prod\frac{ZF}{1}$ sentence of $\mathcal{L}_{\mathfrak{M}}$.

Takahashi [13] gives the following answer to our problem for the case of $\sum_1^{ZF}$ sentences of $\mathcal{L}_{\mathfrak{M}}$:

THEOREM (Takahashi [13]). Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable standard transitive models for ZFC and assume that $\mathfrak{N}$ is an extension of $\mathfrak{M}$ having the same ordinals as $\mathfrak{M}$. If φ is a $\sum_1^{ZF}$ sentence of $\mathcal{L}_{\mathfrak{M}}$ that is true in $\mathfrak{N}$, then there exists a Cohenian extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ that satisfies φ.

Takahashi's proof of his theorem uses a notion of forcing whose conditions are elements of the Lindenbaum algebra of an infinitaly propositional logic.

We shall show that Takahash's theorem may be proved with a very simple notion of forcing. Since in order to do forcing over $\mathfrak{M}$ we need only to be able to code the forcing language and to define the forcing relation in $\mathfrak{M}$, and these do not need the axiom of choice (Cf. Jensen [4]), our proof will improve Takahashi's theorem such, that the theorem applies to models $\mathfrak{M}$ and $\mathfrak{N}$ that do not satisfy the axiom of choice. Also, we shall apply Takahashi's theorem to some $\sum_3^1$ predicates. We will present more applications in a following paper "II".

We assume that the readers are familiar with the notions of first-order languages, formal system of Zermero-Fraenkel set theory in such a language, models for such a system and the analytical hierarchy, and the theory of forcing. The book of Shoenfield [8] provides one of the best accounts of these notions and their theories. For the theory of forcing, the readers should consult the excellent papers of Shenfield [9] and Solovay [12, §1].

Our notations and terminologies are those of Shoenfield[9,10] and Solovay [12,§1], but with the following differences: We use the symbol "$\equiv$" for logical equivalence, and small Greek letters "α", "β" and "γ" denote reals which are totall functions from ω into ω, but letter "σ" is a special variable for ordinals.

<u>1. Shoenfield Absoluteness Theorem.</u> Let us begin with a theorem which is a model theoretic version of well known Shenfield absoluteness theorem([8]). This is considerably important in our further work.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be standard transitive models for <u>ZF</u> and assume that $\mathfrak{N}$ is an extension of $\mathfrak{M}$ having the same countable ordinals as $\mathfrak{M}$. Then we have

<u>THEOREM 1.</u> <u>The Σ^1_2 and Π^1_2 sentences of $\mathcal{L}_{\mathfrak{M}}$ are absolute between $\mathfrak{M}$ and $\mathfrak{N}$</u> .

<u>Proof.</u> Shoenfield [8] shows that if φ is a Σ^1_2 sentence of $\mathcal{L}_{\mathfrak{M}}$ then there is a Δ_1^{ZF} formula $\chi(\sigma)$ having only one free variable σ and the same names as φ such that

(*) $$\varphi \equiv \exists \sigma < \omega_1\ \chi(\sigma)$$

By (*), the absoluteness of the Δ_1^{ZF} sentences(Cf. Karp[5]) and the hypothesis of the theorem,

$$\begin{aligned}\mathfrak{M} \models \varphi &\equiv \mathfrak{M} \models \exists \sigma < \omega_1\ \chi(\sigma)\\ &\equiv \exists \sigma < \omega_1^{\mathfrak{M}} [\ \mathfrak{M} \models \chi(\sigma)\ ]\\ &\equiv \exists \sigma < \omega_1^{\mathfrak{N}} [\ \mathfrak{N} \models \chi(\sigma)\ ]\\ &\equiv \mathfrak{N} \models \exists \sigma < \omega_1\ \chi(\sigma)\end{aligned}$$

$$\equiv \mathfrak{N} \models \varphi .$$

For a Π^1_2 sentence φ , consider the negation of φ which is Σ^1_2.

C.Q.F.D.

2. Main Theorem. Now we turn to our main theorem which is a slight improvement of Takahashi's theorem.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable standard transitive models for ZF and assume that $\mathfrak{N}$ is an extension of $\mathfrak{M}$ having the same ordinals as $\mathfrak{M}$.

THEOREM 2. *If φ is a Σ^{ZF}_1 sentence of $\mathcal{L}_{\mathfrak{N}}$ having only names $\underline{c}_1,\dots,\underline{c}_n$, then there exists a Cohenian extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ that satisfies φ.*

Proof. Without lseing generality, we may assme that φ has only one name $\underline{c}$.

Let $\chi(x,y)$ be a Δ_0 formula of $\mathcal{L}$ having only two free variables x and y such that

$$\varphi \equiv \exists x\, \chi(x,\underline{c})$$

is provable in ZF. Since φ is true in $\mathfrak{N}$, there is a set s of $\mathfrak{N}$ such that $\chi(x,y)$ is satisfied in $\mathfrak{N}$ when x and y are interpleted by s and c respectively. If s is already in $\mathfrak{M}$, then our theorem is trivial. Therefore we may assume that s is not in $\mathfrak{M}$.

Now let us consider two partially ordered structures

$$C_c = (H_{\aleph_0}(\omega, TC(c \cup \omega)), \subseteq)$$

and

$$C_s = (H_{\aleph_0}(\omega, TC(s)), \subseteq).$$

Then C_c and C_s are notions of forcing which are sets in $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Now let G_c and G_s be a $\mathfrak{N}$-generic filter on C_c and a $\mathfrak{N}[G_c]$-generic filter on C_s respectively. Notice that G_c is also $\mathfrak{M}$-generic filter on C_c. Thus there is a bijection g_c from ω onto $TC(c \cup \omega)$ in $\mathfrak{M}[G_c]$ and $\mathfrak{N}[G_c]$. Let g be a bijection from ω onto $TC(c \cup \omega) \cup TC(s)$ in $\mathfrak{N}[G_c, G_s]$ such that for every natural number i,

$$g(2i) = g_c(i).$$

Consider the binary relation R_g on ω defined as follows

$$R_g = \{(i,j) \in \omega \times \omega : g(i) \in g(j)\}.$$

Then g is an isomorphism between two structures (ω, R_g) and $(TC(c \cup \omega) \cup TC(s), \in)$. Let c^* and s^* be two natural numbers such that

$$g(c^*) = c$$

and

$$g(s^*) = s.$$

Since $\chi(x,y)$ is a Δ_0 formula of $\mathcal{L}$, and $(TC(c \cup \omega) \cup TC(s), \in)$ is a transitive substructure of $\mathfrak{N}$,

$$(TC(c \cup \omega) \cup TC(s), \in) \models \chi(x,y)[s,c],$$

and thus

$$(\omega, R_g) \models \chi(x,y)[s^*, c^*].$$

Let $\psi_0(x,y)$ be an arithmetical predicate having only two free variables x and y, without names, which says that x and y are reals such that x and y are the codes of binary relations on ω, and for all natural numbers i and j

$$(*) \qquad x(<2i,2j>) = 0 \equiv y(<2i,2j>) = 0$$

and

$$y(<i,2j+1>) = 1 \,\&\, y(<2i+1,j>) = 1.$$

Let $\psi_1(x)$ be a Π^1_1 predicate having only one free variable x , without names, which says that x is a real that is the code of a well-founded binary relation on ω, and for all natural numbers j and k

$$\forall i(x(<i,j>) = 0 \equiv x(<i,k>) = 0) \rightarrow j = k.$$

Finally let $\psi_2(x)$ be an arithmetical predicate having only one free variable x, without names, that is the logical conjunction of a predicate which says that x is a real and the predicate obtained from the formula $\chi(x, y)$ by replacing x with $\underline{s}^*$, y with $\underline{c}^*$; $u \in v$ with $x(<i,j>) = 0$, $\forall u$ with $\forall i$ and $\exists u$ with $\exists i$. Then

$$\exists x(\psi_0(x,y) \;\&\; \psi_1(x) \;\&\; \psi_2(x))$$

is a Σ^1_2 predicate of $\mathcal{L}_{\mathfrak{M}}$ having only one free real variable y and two names $\underline{c}^*$ and $\underline{s}^*$, and we express this predicate as $\psi(y)$ for simplicity.

Consider the binary relation S_{g_c} defined as follows

$$S_{g_c} = \{(2i,2j) \in \omega \times \omega \;:\; g_c(i) \in g_c(j)\}$$

which is in $\mathfrak{M}[G_c]$ and $\mathfrak{N}[G_c]$.

Let α and β be the codes of R_g and S_{g_c} respectively. Notice β is in $\mathfrak{M}[G_c]$, so in $\mathfrak{N}[G_c]$ and $\mathfrak{N}[G_c,G_s]$. Since R_g is a well-founded binary relation on ω such that for all natural numbers i,j and k

$$(2i,2j) \in R_g \equiv (2i,2j) \in S_{g_c},$$

and

$$(i,2j+1) \notin S_{g_c} \equiv (2i+1,j) \notin S_{g_c}$$

$$\forall n((n,j) \in R_g \equiv (n,k) \in R_g) \rightarrow j = k,$$

and $\psi_2(\alpha)$ says that $\chi(x,y)$ is satisfied in (ω,R_g) when x and y interpreted by s* and c* respectively, we have

$$\mathfrak{N}[G_c,G_s] \models \psi_0(x,y) \;\&\; \psi_1(x) \;\&\; \psi_2(x)\,[\alpha,\beta].$$

thus

$$\mathfrak{N}[G_c,G_s] \models \psi(y)\,[\beta].$$

Now observe that $\mathfrak{M}[G_c]$ is a submodel of $\mathfrak{N}[G_c,G_s]$ having the same countable ordinals as $\mathfrak{N}[G_c,G_s]$, for, since the notions of forcing $(H_{\aleph_0}(\omega,TC(c\cup\omega)),\subseteq)$ and $(H_{\aleph_0}(\omega,TC(s)),\subseteq)$ satisfy the $\aleph_0$-chain condition in $\mathfrak{M}$ and $\mathfrak{N}$ respectively, the cardinals, so the countable ordinals, are preserved between the two models $\mathfrak{M}[G_c]$ and $\mathfrak{N}[G_c,G_s]$. By theorem 1, the Σ^1_2 predicate $\psi(y)$ is also satisfied in $\mathfrak{M}[G_c]$ when y is interpreted by β. Let γ be a real in $\mathfrak{M}[G_c]$ such that

$$\mathfrak{M}[G_c] \models \psi_0(x,y) \;\&\; \psi_1(x) \;\&\; \psi_2(x)\,[\gamma,\beta].$$

Consider the binary relation R_γ defined as follows

$$R_\gamma = \{(i,j) \in \omega \times \omega \quad : \gamma(<i,j>) = 0\}.$$

Then $(\omega,R\ )$ is a well-founded and extensional structure such that

$$(\omega,R_\gamma) \models \chi(x,y)[s^*,c^*].$$

By Mostowski Collapsing Theorem ([7]), there are unique transitive set u and unique isomorphism π from (ω,R_γ) onto $(u,\in)$ in $\mathfrak{M}[G_c]$. Thus

$$(u,\in) \models \chi(x,y)[\pi(s^*),\pi(c^*)].$$

By (*) S_{g_c} is a subset of R_γ, and hence the inverse function of g_c is the restriction of π to the set of even natural numbers. Since c* is a even natural number,

$$\pi(c^*) = g_c^{-1}(c^*) = g^{-1}(c^*) = c.$$

Notice that $(u,\in)$ is a substructure of $\mathfrak{M}[G_c]$, and $\chi(x,y)$ is a Δ_0 formula of $\mathcal{L}$. Therefore we have

$$\mathfrak{M}[G_c] \models \chi(x,y)[\pi(s^*),c],$$

so

$$\mathfrak{M}[G_c] \models \varphi.$$

C.Q.F.D.

3. Application. Now we shall give some applications of our theorem 2 that are concerned with the analytical hierarchy.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable transitive models for ZF + "there existe an inaccessible cardinal" and assume that $\mathfrak{N}$ is an extension of $\mathfrak{M}$ having the same ordinals. Let $\exists\beta\varphi(\underset{\sim}{\alpha},\beta)$ be a Σ^1_3 sentence with one name $\underset{\sim}{\alpha}$ for a real in $\mathfrak{M}$.

THEOREM 3. If the predicate $\exists\beta\varphi(\underline{\alpha},\beta)$ is satisfied in $\mathfrak{N}$, then there exists a Cohenian extension $\mathfrak{M}[G]$ which has a standard transitive submodel for ZF + $\exists\beta\varphi(\underline{\alpha},\beta)$.

Proof. Let σ be an inaccessible cardinal in $\mathfrak{N}$ and $R^{\mathfrak{N}}(\sigma)$ the set of sets in $\mathfrak{N}$ with ranks less than σ. Let $\psi(x,y,\underline{\alpha})$ be the Δ_1^{ZF} formula "$(x,\in)$ is a transitive model for ZF + $\varphi(\underline{\alpha},y)$". Then $\psi(x,y,\underline{\alpha})$ is satisfied in $\mathfrak{N}$ when x and y are intrepleted by $R^{\mathfrak{N}}(\sigma)$ and some real in $R^{\mathfrak{N}}(\sigma)$ respectively. Since $\exists x \exists y \psi(x,y,\underline{\alpha})$ is a Σ_1^{ZF} formula, by our theorem 2 there exists a Cohenian extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ in which this formula is true. This means that there is a transitive standard submodel of $\mathfrak{M}[G]$ in which $\exists\beta\varphi(\underline{\alpha},\beta)$ is true. C.Q.F.D.

The technique in the proof of theorem 3 has some interest and many applications, and we present here one more application.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be standard transitive models for ZF having the same ordinals and assume that $\mathfrak{N}$ satisfies MC(there exists a mesurable crdinal). Let $P(\alpha,\beta)$ be a Π_2^1 predicate which says that $\beta = \alpha^{\#}$(Cf. Solovay [11]). Since $\exists\beta P(\underline{\alpha},\beta)$ is provable in ZF + MC, for each real α in $\mathfrak{M}$, $\exists\beta P(\underline{\alpha},\beta)$ is true in $\mathfrak{N}$. Let $\psi(x,y,\underline{\alpha})$ be a Δ_1^{ZF} formula which says that $(x,\in)$ is a transitive model of ZF + $P(\underline{\alpha},y)$. Then applying a similar argument in the proof of theorem 3 to this formula, we have

THEOREM 4. There exists a Cohenian extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ which has a standard transitive submodel for ZF + "$\underline{\alpha}^{\#}$ exists".

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